

Noncommutative Deformations of Locally Symmetric Kähler manifolds

¹ Kentaro Hara and ² Akifumi Sako

^{1,2} Department of Mathematics, Faculty of Science Division II,
Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

² Fakultät für Physik, Universität Wien
Boltzmanngasse 5, A-1090 Wien, Austria

Abstract

We derive algebraic recurrence relations to obtain a deformation quantization with separation of variables for a locally symmetric Kähler manifold. This quantization method is one of the ways to perform a deformation quantization of Kähler manifolds, which is introduced by Karabegov. From the recurrence relations, concrete expressions of star products for one-dimensional local symmetric Kähler manifolds and $\mathbb{C}P^N$ are constructed. The recurrence relations for a Grassmann manifold $G_{2,2}$ are closely studied too.

1 Introduction

Deformation quantizations were introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [3] as a method to quantize spaces. After [3], several ways of deformation quantization were proposed [8, 26, 9, 20]. In particular, deformation quantizations of Kähler manifolds were provided in [22, 23, 6, 7]. The deformation quantization with separation of variables is one of the methods to construct noncommutative Kähler manifolds given by Karabegov [14, 15, 16]. In this article, only the deformation quantization with separation of variables is studied.

A noncommutative product on a quantized manifold is given by a star product which is defined in a form of a formal power series of deformation parameter \hbar . The star product is obtained as solutions of an infinite system of differential equations. The existence of the solution is proved for a wide class of manifolds, however explicit expressions of deformation quantizations are constructed only for few kinds of manifolds, because it is difficult to obtain a concrete solution of the system of differential equations, in general.

In [27] explicit expressions of star products of noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$ were provided as the deformation quantization with separation of variables. ¹ The reason that they were provided is that $\mathbb{C}P^N$ and $\mathbb{C}H^N$ are locally symmetric spaces. The symmetry makes our problems be simple ones. In [27], star products on general locally symmetric Kähler manifolds are also

¹See also the following references. Star products on the fuzzy $\mathbb{C}P^N$ are investigated in [1, 18, 13]. A deformation quantization of the hyperbolic plane was provided in [4].

discussed, but it is not enough to obtain explicit expression of the star products.

The purpose of this article is to derive algebraic recurrence relations to make concrete expression of star products on locally symmetric Kähler manifolds. The method in this article overcome the problems in [27]. Using the recurrence relations, star products on some locally symmetric Kähler manifolds, the Riemann surfaces and $\mathbb{C}P^N$, are obtained.

The organization of this article is as follows. In Section 2, we review the deformation quantization with separation of variables proposed by Karabegov. In Section 3, explicit formulas to obtain star products on local symmetric Kähler manifolds are given explicitly. In Section 4, the explicit expression of star products of one-dimensional locally symmetric Kähler manifolds (Riemann surfaces) are constructed. A two-dimensional case is also discussed. In Section 5, we give an explicit formula to obtain star products on a Grassmann manifold, and star products on $\mathbb{C}P^N$ are obtained by using the formula. Finally, we summarize our results and discuss their several perspectives in Section 6.

2 Review of the deformation quantization with separation of variables

In this section, we review the deformation quantization with separation of variables to construct noncommutative Kähler manifolds.

An N -dimensional Kähler manifold M is described by using a Kähler potential. Let Φ be a Kähler potential and ω be a Kähler 2-form:

$$\omega := ig_{k\bar{l}}dz^k \wedge d\bar{z}^l, \quad g_{k\bar{l}} := \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}. \quad (2.1)$$

where z^i, \bar{z}^i ($i = 1, 2, \dots, N$) are complex local coordinates.

In this article, we use the Einstein summation convention over repeated indices. The $g^{\bar{k}l}$ is the inverse of the Kähler metric tensor $g_{k\bar{l}}$. That means $g^{\bar{k}l}g_{l\bar{m}} = \delta_{\bar{k}\bar{m}}$. In the following, we use

$$\partial_k = \frac{\partial}{\partial z^k}, \quad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}. \quad (2.2)$$

Deformation quantization is defined as follows.

Definition 1 (Deformation quantization). Deformation quantization of Poisson manifolds is defined as follows. \mathcal{F} is defined as a set of formal power series: $\mathcal{F} := \left\{ f \mid f = \sum_k f_k \hbar^k, f_k \in C^\infty(M) \right\}$. A star product is defined as

$$f * g = \sum_k C_k(f, g) \hbar^k \quad (2.3)$$

such that the product satisfies the following conditions.

1. $(\mathcal{F}, +, *)$ is a (noncommutative) algebra.
2. $C_k(\cdot, \cdot)$ is a bidifferential operator.
3. C_0 and C_1 are defined as

$$C_0(f, g) = fg, \quad (2.4)$$

$$C_1(f, g) - C_1(g, f) = \{f, g\}, \quad (2.5)$$

where $\{f, g\}$ is the Poisson bracket.

4. $f * 1 = 1 * f = f$.

Karabegov introduced a method to obtain a deformation quantization of a Kähler manifold in [15]. His deformation quantization is called deformation quantizations with separation of variables

Definition 2 (A star product with separation of variables). $*$ is called a star product with separation of variables on a Kähler manifold when

$$a * f = af \quad (2.6)$$

for an arbitrary holomorphic function a and

$$f * b = fb \quad (2.7)$$

for an arbitrary anti-holomorphic function b .

We use

$$D^{\bar{l}} = g^{\bar{l}k} \partial_k$$

and introduce

$$\mathcal{S} := \left\{ A \mid A = \sum_{\alpha} a_{\alpha} D^{\alpha}, \quad a_{\alpha} \in C^{\infty}(M) \right\},$$

where α is a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. In this article, we also use the Einstein summation convention over repeated multi-indices and $a_{\alpha} D^{\alpha} := \sum_{\alpha} a_{\alpha} D^{\alpha}$.

There are some useful formulae. $D^{\bar{l}}$ satisfies the following equations.

$$[D^{\bar{l}}, D^{\bar{m}}] = 0 \quad , \quad [D^{\bar{l}}, \partial_{\bar{m}} \Phi] = \delta^{\bar{l}}_{\bar{m}}, \quad \forall l, m, \quad (2.8)$$

where $[A, B] = AB - BA$. Using them, one can construct a star product as a differential operator L_f such that $f * g = L_f g$.

Theorem 2.1. [Karabegov [15]]. For an arbitrary Kähler form ω , there exist a star product with separation of variables $*$ and it is constructed as follows. Let f be an element of \mathcal{F} and $A_n \in \mathcal{S}$ be a differential operator whose coefficients depend on f i.e.

$$A_n = a_{n,\alpha}(f)D^\alpha, \quad D^\alpha = \prod_{i=1}^n (D^{\bar{i}})^{\alpha_i}, \quad (D^{\bar{i}}) = g^{\bar{i}l}\partial_l, \quad (2.9)$$

where α is a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. Then,

$$L_f = \sum_{n=0}^{\infty} \hbar^n A_n \quad (2.10)$$

is uniquely determined such that it satisfies the following conditions.

1. For $R_{\partial_{\bar{l}}\Phi} = \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}$,

$$[L_f, R_{\partial_{\bar{l}}\Phi}] = 0. \quad (2.11)$$

2.

$$L_f 1 = f * 1 = f. \quad (2.12)$$

Then the star products are given by

$$L_f g := f * g, \quad (2.13)$$

and the star products satisfy the associativity;

$$L_h(L_g f) = h * (g * f) = (h * g) * f = L_{L_h g} f. \quad (2.14)$$

Recall that each two of $D^{\bar{i}}$ commute each other, so if a multi index α is fixed then the A_n is uniquely determined. (2.12)-(2.14) imply that $L_f g = f * g$ gives deformation quantization.

Definition 3. A map from differential operators to formal polynomials is defined as

$$\sigma(A; \xi) := \sum_{\alpha} a_{\alpha} \xi^{\alpha},$$

where

$$A = \sum_{\alpha} a_{\alpha} D^{\alpha}.$$

This map is called “twisted symbol”. It becomes easier to calculate commutators by using the following theorem.

Proposition 2.2 (Karabegov [15]). *Let $a(\xi)$ be a twisted symbol of an operator A . Then the twisted symbol of the operator $[A, \partial_{\bar{i}}\Phi]$ is equal to $\partial a / \partial \xi^{\bar{i}}$;*

$$\sigma([A, \partial_{\bar{i}}\Phi]) = \frac{\partial}{\partial \xi^{\bar{i}}} \sigma(A).$$

This proposition follows from From (2.8), i.e.

$$\sigma([D^{\bar{l}}, \partial_{\bar{i}}\Phi]) = \delta_{\bar{i}}^{\bar{l}}.$$

3 Deformation quantization with separation of variables for a locally symmetric Kähler manifold

In this section, explicit formulas to obtain star products on local symmetric Kähler manifolds are constructed. A method of Karabegov in Section 2 is used for the constructing.

At first we list notations used in this article. Let M be a N -dimensional Kähler manifold, $\partial_i := \frac{\partial}{\partial z_i}, \partial_{\bar{i}} := \frac{\partial}{\partial \bar{z}_i}$ ($i = 1, \dots, N$) be tangent vector fields on a coordinate chart $U \subset M$ with its local coordinates $(z^1, \dots, z^N, \bar{z}^1, \dots, \bar{z}^N)$, $dz^i, d\bar{z}^i$ be cotangent vector fields on U and $Y^{\mu_1 \dots \mu_k \bar{\mu}_1 \dots \bar{\mu}_m}{}_{\nu_1 \dots \nu_l \bar{\nu}_1 \dots \bar{\nu}_n} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_k} \otimes \partial_{\bar{\mu}_1} \otimes \dots \otimes \partial_{\bar{\mu}_m} \otimes dz^{\nu_1} \otimes \dots \otimes dz^{\nu_l} \otimes d\bar{z}^{\bar{\nu}_1} \otimes \dots \otimes d\bar{z}^{\bar{\nu}_n} \in \Gamma[(T^{1,0}M)^{\otimes k} \otimes (T^{0,1}M)^{\otimes m} \otimes \{(T^{1,0}M)^*\}^{\otimes l} \otimes \{(T^{0,1}M)^*\}^{\otimes n}]$ be a $((k, m), (l, n))$ -tensor field. The classical style of covariant derivative $\nabla_i := \nabla_{\partial_i}$ acts on coefficients of tensor fields as

$$\begin{aligned} \nabla_i Y^{\mu_1 \dots \mu_k \bar{\mu}_1 \dots \bar{\mu}_m}{}_{\nu_1 \dots \nu_l \bar{\nu}_1 \dots \bar{\nu}_n} &= \partial_i Y^{\mu_1 \dots \mu_k \bar{\mu}_1 \dots \bar{\mu}_m}{}_{\nu_1 \dots \nu_l \bar{\nu}_1 \dots \bar{\nu}_n} \\ &+ \sum_{q=1}^k \Gamma_{i\rho_q}^{\mu_q} Y^{\mu_1 \mu_2 \dots \rho_q \dots \mu_k \bar{\mu}_1 \dots \bar{\mu}_m}{}_{\nu_1 \dots \nu_l \bar{\nu}_1 \dots \bar{\nu}_n} + \sum_{q=1}^m \Gamma_{i\bar{\rho}_q}^{\bar{\mu}_q} Y^{\mu_1 \dots \mu_k \bar{\mu}_1 \bar{\mu}_2 \dots \bar{\rho}_q \dots \bar{\mu}_m}{}_{\nu_1 \dots \nu_l \bar{\nu}_1 \dots \bar{\nu}_n} \\ &- \sum_{q=1}^l \Gamma_{i\nu_q}^{\sigma_q} Y^{\mu_1 \dots \mu_k \bar{\mu}_1 \dots \bar{\mu}_m}{}_{\nu_1 \nu_2 \dots \sigma_q \dots \nu_l \bar{\nu}_1 \dots \bar{\nu}_n} - \sum_{q=1}^n \Gamma_{i\bar{\nu}_q}^{\bar{\sigma}_q} Y^{\mu_1 \dots \mu_k \bar{\mu}_1 \dots \bar{\mu}_m}{}_{\nu_1 \dots \nu_l \bar{\nu}_1 \bar{\nu}_2 \dots \bar{\sigma}_q \dots \bar{\nu}_n} \end{aligned}$$

where Γ_{jk}^i is the Christoffel symbol.

The Riemannian curvature of a Hermitian manifold M is defined as

$$R_{i\bar{j}k}{}^l = \partial_i \Gamma_{\bar{j}k}^l - \partial_{\bar{j}} \Gamma_{ik}^l + \Gamma_{jk}^n \Gamma_{in}^l - \Gamma_{ik}^n \Gamma_{jn}^l.$$

For Hermitian manifolds, the Christoffel symbols are given as

$$\Gamma_{jk}^l = g^{l\bar{q}} \frac{\partial g_{j\bar{q}}}{\partial z^k}.$$

The Riemannian curvature of a Hermitian manifold M is obtained as

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{\bar{j}i}}{\partial z^k \partial \bar{z}^l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{\bar{j}p}}{\partial \bar{z}^l}.$$

On a Kähler manifold, its metric is described by using Kähler potential Φ as (2.1). Then its Riemannian curvature is given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^4 \Phi}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l} + g^{p\bar{q}} \frac{\partial^3 \Phi}{\partial z^i \partial \bar{z}^q \partial z^k} \frac{\partial^3 \Phi}{\partial z^p \partial \bar{z}^j \partial \bar{z}^l}. \quad (3.1)$$

(See [19] P157.)

Operators $D^{\vec{\alpha}_n}$ and $D^{\vec{\beta}_n^*}$ are defined by using $D^k = g^{k\bar{m}} \partial_{\bar{m}}$ and $D^{\bar{j}} = g^{\bar{j}l} \partial_l$ as

$$D^{\vec{\alpha}_n} := D^{\alpha_1^n} D^{\alpha_2^n} \dots D^{\alpha_N^n}, \quad D^{\vec{\beta}_n} := D^{\beta_1} D^{\beta_2} \dots D^{\beta_N}$$

where

$$D^{\alpha_k} := \left(D^k\right)^{\alpha_k}, \quad D^{\beta_j} := \left(D^{\bar{j}}\right)^{\beta_j},$$

and $\vec{\alpha}_n$ and $\vec{\beta}_n^*$ are N -dimensional vectors whose summation of their all elements are set to be n ;

$$\vec{\alpha}_n \in \left\{ (\gamma_1^n, \gamma_2^n, \dots, \gamma_N^n) \in \mathbb{Z}^N \mid \sum_{k=1}^N \gamma_k^n = n \right\}, \quad \vec{\beta}_n^* \in \left\{ (\gamma_1^n, \gamma_2^n, \dots, \gamma_N^n)^* \in \mathbb{Z}^N \mid \sum_{k=1}^N \gamma_k^n = n \right\}$$

i.e.

$$\begin{aligned} \vec{\alpha}_n &:= (\alpha_1^n, \alpha_2^n, \dots, \alpha_N^n), \quad |\vec{\alpha}_n| := \sum_{k=1}^N \alpha_k^n = n \\ \vec{\beta}_n^* &:= (\beta_1^n, \beta_2^n, \dots, \beta_N^n)^*, \quad |\vec{\beta}_n^*| := \sum_{k=1}^N \beta_k^n = n. \end{aligned}$$

For $\vec{\alpha}_n \notin \mathbb{Z}_{\geq 0}^N$ we define $D^{\vec{\alpha}_n} := 0$.

For example, $D^{(1,2,3)} = D^1 (D^2)^2 (D^3)^3$, $D^{(2,4,0)^*} = (D^{\bar{1}})^2 (D^{\bar{2}})^4$ and $D^{(5,-2,3)} = 0$ for a 3-dimensional manifolds case with $n = 6$.

\vec{e}_i is used as a N -dimensional vector

$$\vec{e}_i = (\delta_{1i}, \delta_{2i}, \dots, \delta_{Ni}). \quad (3.2)$$

From here to the end of this section, we make up recurrence relations to construct explicit expressions of star products on locally symmetric Kähler manifolds.

A Riemannian(Kähler) manifold (M, g) is called a locally symmetric Riemannian(Kähler) manifold when $\nabla_m R_{ijk}{}^l = 0$ ($\forall i, j, k, l, m$). Only locally symmetric Kähler manifolds are studied in this article.

We assume that a star product with separation of variables for smooth functions f and g on a locally symmetric Kähler manifold M has a form

$$L_f g = f * g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^*} g \right), \quad (3.3)$$

where $T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n$ are covariantly constants. If $\vec{\alpha}_n \notin \mathbb{Z}_{\geq 0}^N$ or $\vec{\beta}_n \notin \mathbb{Z}_{\geq 0}^N$ then we define $T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n := 0$. $\sum_{\vec{\alpha}_n, \vec{\beta}_n^*}$ is defined by the summation over all $\vec{\alpha}_n^*$ and $\vec{\beta}_n^*$ satisfying $|\vec{\alpha}_n^*| = |\vec{\beta}_n^*| = n$. In brief,

$$n = |\vec{\alpha}_n^*| := \sum_{i=1}^N \alpha_i^n, \quad n = |\vec{\beta}_n^*| := \sum_{i=1}^N \beta_i^n, \quad \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} := \sum_{|\vec{\alpha}_n| = |\vec{\beta}_n| = n}.$$

Proposition 3.1. *For the star product on a locally symmetric Kähler manifold M as (3.3), $T_{\vec{\alpha}_0, \vec{\beta}_0^*}^0$ and $T_{\vec{e}_i, \vec{e}_j}^1$ are given as*

$$T_{\vec{\alpha}_0, \vec{\beta}_0^*}^0 = 1, \quad T_{\vec{e}_i, \vec{e}_j}^1 = \hbar g_{i\bar{j}}.$$

Proof. From (3.3), the star product for smooth functions f and g on M is given as

$$L_f g = T_{\vec{\alpha}_0, \vec{\beta}_0^*}^0 f g + \sum_{\vec{\alpha}_1, \vec{\beta}_1^*} T_{\vec{\alpha}_1, \vec{\beta}_1^*}^1 \left(D^{\vec{\alpha}_1} f \right) \left(D^{\vec{\beta}_1^*} g \right) + \sum_{n=2}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^*} g \right).$$

$T_{\vec{\alpha}_0, \vec{\beta}_0^*}^0 = 1$ is trivial. $C_1(f, g)$ is expressed as

$$C_1(f, g) = \sum_{\vec{\alpha}_1, \vec{\beta}_1^*} T_{\vec{\alpha}_1, \vec{\beta}_1^*}^1 \left(D^{\vec{\alpha}_1} f \right) \left(D^{\vec{\beta}_1^*} g \right).$$

By the definition of the deformation quantization (2.5) the first term is related to the Poisson bracket:

$$\begin{aligned} & \hbar \sum_{i, j=1}^n g^{i\bar{j}} \left(\frac{\partial f}{\partial z^i} \frac{\partial g}{\partial \bar{z}^j} - \frac{\partial g}{\partial \bar{z}^j} \frac{\partial f}{\partial z^i} \right) \\ &= \sum_{\vec{\alpha}_1, \vec{\beta}_1^*} (T_{\vec{\alpha}_1, \vec{\beta}_1^*}^1 (g^{\alpha_1 \bar{m}} \partial_{\bar{m}} f) (g^{\bar{\beta}_1 l} \partial_l g) - T_{\vec{\alpha}_1, \vec{\beta}_1^*}^1 (g^{\alpha_1 \bar{m}} \partial_{\bar{m}} g) (g^{\bar{\beta}_1 l} \partial_l f)). \end{aligned}$$

Then $T_{\vec{e}_i, \vec{e}_j}^1 = \hbar g_{i\bar{j}}$ is shown. □

The purpose of remained part of this section is to replace the recurrence relations as differential equations by those of algebraic equations. We need to calculate $[L_f, \partial_i \Phi]$ and $[L_f, \partial_{\bar{i}}]$ in (2.11).

Proposition 3.2. *Let f and g be smooth functions on a locally symmetric Kähler manifold M and L_f be a left star product by f given as (3.3). Then*

$$\begin{aligned} \sigma([L_f, \partial_i \Phi]) &= \frac{\partial \sigma(L_f)}{\partial \xi^{\bar{i}}} \\ &= \begin{cases} \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} \beta_i^n T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n} f) (\xi^{\bar{1}\beta_1^n} \dots \xi^{\bar{i}\beta_i^n-1} \dots \xi^{\bar{N}\beta_N^n}) & (\beta_i \neq 0) \\ 0 & (\beta_i^n = 0) \end{cases}, \end{aligned}$$

or equivalently,

$$[L_f, \partial_i \Phi] g = \begin{cases} \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} \beta_i^n T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n} f) (D^{\vec{\beta}_n^* - \vec{e}_i} g) & (\beta_i \neq 0) \\ 0 & (\beta_i^n = 0) \end{cases}. \quad (3.4)$$

Proof. By Proposition 2.2,

$$\sigma([L_f, \partial_i \Phi]) = \frac{\partial \sigma(L_f)}{\partial \xi^{\bar{i}}} = \frac{\partial}{\partial \xi^{\bar{i}}} \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n} f) (\xi^{\vec{\beta}_n^*}).$$

$\xi^{\vec{\beta}_n^*}$ is explicitly written by $\xi^{\vec{\beta}_n^*} = \xi^{\bar{1}\beta_1^n} \xi^{\bar{2}\beta_2^n} \dots \xi^{\bar{N}\beta_N^n}$, then

$$\begin{aligned} \sigma([L_f, \partial_i \Phi]) &= \begin{cases} \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} \beta_i^n T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n} f) (\xi^{\bar{1}\beta_1^n} \dots \xi^{\bar{i}\beta_i^n-1} \dots \xi^{\bar{N}\beta_N^n}) & (\beta_i \neq 0) \\ 0 & (\beta_i = 0) \end{cases}. \end{aligned}$$

□

The following formulas are given in [28].

Fact 3.3. *For smooth functions f and g on a locally symmetric Kähler manifold, the following formulas are given.*

$$\begin{aligned} \nabla_{\bar{j}_1} \dots \nabla_{\bar{j}_n} f &= g_{l_1 \bar{j}_1} \dots g_{l_n \bar{j}_n} D^{l_1} \dots D^{l_n} f \\ \nabla_{k_1} \dots \nabla_{k_n} g &= g_{\bar{m}_1 k_1} \dots g_{\bar{m}_n k_n} D^{\bar{m}_1} \dots D^{\bar{m}_n} g \\ D^{l_1} \dots D^{l_n} f &= g^{l_1 \bar{j}_1} \dots g^{l_n \bar{j}_n} \nabla_{\bar{j}_1} \dots \nabla_{\bar{j}_n} f \\ D^{\bar{m}_1} \dots D^{\bar{m}_n} g &= g^{\bar{m}_1 k_1} \dots g^{\bar{m}_n k_n} \nabla_{k_1} \dots \nabla_{k_n} g. \end{aligned}$$

Fact 3.3 derives the following lemma.

Lemma 3.4. *Let f and g be smooth functions on a locally symmetric Kähler manifold M . Let L_f be a left star product by f given as (3.3). Then,*

$$\begin{aligned}
& [L_f, \hbar \partial_{\bar{i}}]g \\
&= \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \sum_{k=1}^N \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \frac{\beta_k^n (\beta_k^n - 1)}{2} R_{\bar{\rho}}^{\bar{k} \bar{k}} \bar{i} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^* + \vec{e}_{\bar{\rho}} - \vec{e}_{\bar{k}}} g \right) \\
&+ \hbar \sum_{n=0}^{\infty} \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \beta_k^n \beta_{k+l}^n R_{\bar{\rho}}^{\bar{k} + \bar{l} \bar{k}} \bar{i} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^* + \vec{e}_{\bar{\rho}} - \vec{e}_{\bar{k}}} g \right) \\
&- \hbar \sum_{n=1}^{\infty} \sum_{\vec{\alpha}_{n-1} \vec{\beta}_{n-1}^*} \sum_{d=1}^N g_{id} T_{\vec{\alpha}_{n-1} \vec{\beta}_{n-1}^*}^{n-1} \left(D^{\vec{\alpha}_{n-1} + \vec{e}_{\bar{d}}} f \right) \left(D^{\vec{\beta}_{n-1}^*} g \right).
\end{aligned}$$

Proof. We can calculate $[L_f, \hbar \partial_{\bar{i}}]g$ straightforwardly.

$$\begin{aligned}
& [L_f, \hbar \partial_{\bar{i}}]g = \hbar [L_f, \nabla_{\bar{i}}]g \\
&= \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left\{ (D^1)^{\alpha_1^n} \dots (D^N)^{\alpha_N^n} f \right\} \left[(D^{\bar{1}})^{\beta_1^n} \dots (D^{\bar{N}})^{\beta_N^n}, \nabla_{\bar{i}} \right] g \\
&- \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left\{ \nabla_{\bar{i}} (D^1)^{\alpha_1^n} \dots (D^N)^{\alpha_N^n} f \right\} \left\{ (D^{\bar{1}})^{\beta_1^n} \dots (D^{\bar{N}})^{\beta_N^n} g \right\}.
\end{aligned} \tag{3.5}$$

From Fact 3.3, the second term of (3.5) becomes

$$\begin{aligned}
& \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left\{ \nabla_{\bar{i}} (D^1)^{\alpha_1^n} \dots (D^N)^{\alpha_N^n} f \right\} \left\{ (D^{\bar{1}})^{\beta_1^n} \dots (D^{\bar{N}})^{\beta_N^n} g \right\} \\
&= \hbar \sum_{n=1}^{\infty} \sum_{d=1}^N \sum_{\vec{\alpha}_{n-1} \vec{\beta}_{n-1}^*} g_{id} T_{\vec{\alpha}_{n-1} \vec{\beta}_{n-1}^*}^{n-1} \left(D^{\vec{\alpha}_{n-1} + \vec{e}_{\bar{d}}} f \right) \left(D^{\vec{\beta}_{n-1}^*} g \right).
\end{aligned} \tag{3.6}$$

To calculate the first term of (3.5) we calculate $[D^{\vec{\beta}_n^*}, \nabla_{\bar{i}}]g$:

$$\begin{aligned}
& \left[(D^{\bar{1}})^{\beta_1^n} \dots (D^{\bar{N}})^{\beta_N^n}, \nabla_{\bar{i}} \right] g \\
&= \left\{ \left[(D^{\bar{1}})^{\beta_1^n}, \nabla_{\bar{i}} \right] \left\{ (D^{\bar{2}})^{\beta_2^n} \dots (D^{\bar{N}})^{\beta_N^n} \right\} + \dots + \left\{ (D^{\bar{1}})^{\beta_1^n} \dots (D^{\bar{N}-1})^{\beta_{N-1}^n} \right\} \left[(D^{\bar{N}})^{\beta_N^n}, \nabla_{\bar{i}} \right] \right\} g.
\end{aligned}$$

For these terms, we evaluate

$$\left[(D^{\bar{a}})^{\beta_a^n}, \nabla_{\bar{i}} \right] \left(D^{\overline{a+1}} \right)^{\beta_{a+1}^n} \dots \left(D^{\bar{N}} \right)^{\beta_N^n} g = \sum_{m=1}^{\beta_a^n} (D^{\bar{a}})^{m-1} [D^{\bar{a}}, \nabla_{\bar{i}}] (D^{\bar{a}})^{\beta_a^n - m} \left(D^{\overline{a+1}} \right)^{\beta_{a+1}^n} \dots \left(D^{\bar{N}} \right)^{\beta_N^n} g \quad (3.7)$$

by cases, Case1 $\beta_a^n = 1$, Ace2 $\beta_a^n > 1$ and $\sum_{k=a+1}^N \beta_k^n > 0$, and Cave3 $\beta_a^n > 1$ and $\sum_{k=a+1}^N \beta_k^n = 0$.

Cache1. If $\beta_a^n = 1$ the last line of (3.7) is written as

$$\begin{aligned} & \left[(D^{\bar{a}})^{\beta_a^n}, \nabla_{\bar{i}} \right] \left(D^{\overline{a+1}} \right)^{\beta_{a+1}^n} \dots \left(D^{\bar{N}} \right)^{\beta_N^n} g \\ &= \sum_{j=a+1}^N \sum_{n_j=1}^{\beta_j^n} R_{\bar{i} \bar{a} \bar{j} \bar{c}} D^{\bar{c}} \left(D^{\overline{a+1}} \right)^{\beta_{a+1}^n} \dots \left(D^{\bar{j}} \right)^{\beta_j^n - 1} \dots \left(D^{\bar{N}} \right)^{\beta_N^n} g \\ &= \sum_{j=a+1}^N \beta_j^n \beta_a^n R_{\bar{i} \bar{a} \bar{j} \bar{c}} D^{\bar{c}} (D^{\bar{a}})^{\beta_a^n - 1} \left(D^{\overline{a+1}} \right)^{\beta_{a+1}^n} \dots \left(D^{\bar{j}} \right)^{\beta_j^n - 1} \dots \left(D^{\bar{N}} \right)^{\beta_N^n} g. \end{aligned} \quad (3.8)$$

Recall that $(D^{\bar{j}})^n = 0$ for negative n by definition.

Carce2. If $\beta_a^n > 1$ and $\sum_{k=a+1}^N \beta_k^n > 0$, by using Fact 3.3, we obtain (3.7) as

$$\begin{aligned} & \sum_{m=1}^{\beta_a^n} (D^{\bar{a}})^{m-1} [D^{\bar{a}}, \nabla_{\bar{i}}] (D^{\bar{a}})^{\beta_a^n - m} \left(D^{\overline{a+1}} \right)^{\beta_{a+1}^n} \dots \left(D^{\bar{N}} \right)^{\beta_N^n} g \\ &= \sum_{m=1}^{\beta_a^n} (D^{\bar{a}})^{m-1} g^{\bar{a}b} g^{\bar{a}k_{a,1}} \dots g^{\bar{a}k_{a,\beta_a^n - m}} [\nabla_b, \nabla_{\bar{i}}] \nabla_{k_{a,1}} \dots \nabla_{k_{a,\beta_a^n - m}} \left(D^{\overline{a+1}} \right)^{\beta_{a+1}^n} \dots \left(D^{\bar{N}} \right)^{\beta_N^n} g \\ &= \sum_{m=1}^{\beta_a^n} \sum_{n_a=1}^{\beta_a^n - m} (D^{\bar{a}})^{m-1} R_{\bar{i} \bar{a} \bar{a} \bar{c}} D^{\bar{c}} (D^{\bar{a}})^{\beta_a^n - m} \left(D^{\overline{a+1}} \right)^{\beta_{a+1}^n} \dots \left(D^{\bar{N}} \right)^{\beta_N^n} g \\ &+ \sum_{m=1}^{\beta_a^n} \sum_{j=a+1}^N \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \beta_j^n R_{\bar{i} \bar{a} \bar{j} \bar{c}} D^{\bar{c}} (D^{\bar{a}})^m \left(D^{\overline{a+1}} \right)^{\beta_{a+1}^n} \dots \left(D^{\bar{j}} \right)^{\beta_j^n - 1} \dots \left(D^{\bar{N}} \right)^{\beta_N^n} g. \end{aligned} \quad (3.10)$$

Here, we used

$$[\nabla_i, \nabla_j] \nabla_{k_1} \dots \nabla_{k_m} f = - \sum_{n=1}^m R_{ijk_n}{}^l \nabla_{k_1} \dots \nabla_{k_{n-1}} \nabla_l \nabla_{k_{n+1}} \dots \nabla_{k_m} f. \quad (3.11)$$

for $m \geq 1$.

Carce3. If $\sum_{k=a+1}^N \beta_k^n = 0$ the R.H.S of (3.7) is written as

$$\sum_{m=1}^{\beta_a^n} (D^{\bar{a}})^{m-1} [D^{\bar{a}}, \nabla_{\bar{i}}] (D^{\bar{a}})^{\beta_a^n - m} g = \sum_{m=1}^{\beta_a^n} \sum_{n_a=1}^{\beta_a^n - m} (D^{\bar{a}})^{m-1} R_{\bar{i} \bar{a} \bar{c}}^{\bar{a} \bar{a}} D^{\bar{c}} (D^{\bar{a}})^{\beta_a^n - m - 1} g. \quad (3.12)$$

Putting Carce 1,2 and 3 into a shape and recalling that M is a locally symmetric Kähler manifold, (3.7) is rewritten as

$$\begin{aligned} & \frac{\beta_a^n (\beta_a^n - 1)}{2} R_{\bar{i} \bar{a} \bar{c}}^{\bar{a} \bar{a}} D^{\bar{c}} (D^{\bar{a}})^{\beta_a^n - 2} (D^{\bar{a}+1})^{\beta_{a+1}^n} \dots (D^{\bar{N}})^{\beta_N^n} g \\ & + \sum_{j=a+1}^N \beta_j^n \beta_a^n R_{\bar{i} \bar{a} \bar{c}}^{\bar{a} \bar{j}} D^{\bar{c}} (D^{\bar{a}})^{\beta_a^n - 1} (D^{\bar{a}+1})^{\beta_{a+1}^n} \dots (D^{\bar{j}})^{\beta_j^n - 1} \dots (D^{\bar{N}})^{\beta_N^n} g. \end{aligned} \quad (3.13)$$

Then we find that the first term of (3.5) is expressed as

$$\begin{aligned} & \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \sum_{k=1}^N \frac{\beta_k^n (\beta_k^n - 1)}{2} R_{\bar{c} \bar{k} \bar{i}}^{\bar{k} \bar{k}} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n} f) (D^{\vec{\beta}_n^* + \vec{e}_c - 2\vec{e}_k} g) \\ & + \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \sum_{k=1}^N \sum_{l=1}^{N-k} \beta_k^n \beta_{k+l}^n R_{\bar{c} \bar{k} \bar{i}}^{\bar{k} + \bar{l} \bar{k}} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n} f) (D^{\vec{\beta}_n^* + \vec{e}_c - \vec{e}_k - \vec{e}_{k+l}} g) \end{aligned} \quad (3.14)$$

Finally, we get the result with substituting (3.6) and (3.14) into (3.5)

$$\begin{aligned} [L_f, \hbar \partial_{\bar{i}}] g &= \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \sum_{k=1}^N \frac{\beta_k^n (\beta_k^n - 1)}{2} R_{\bar{c} \bar{k} \bar{i}}^{\bar{k} \bar{k}} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n} f) (D^{\vec{\beta}_n^* + \vec{e}_c - 2\vec{e}_k} g) \\ & + \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \sum_{k=1}^N \sum_{l=1}^{N-k} \beta_k^n \beta_{k+l}^n R_{\bar{c} \bar{k} \bar{i}}^{\bar{k} + \bar{l} \bar{k}} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n} f) (D^{\vec{\beta}_n^* + \vec{e}_c - \vec{e}_k - \vec{e}_{k+l}} g) \\ & - \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \sum_{d=1}^N g_{\bar{i}d} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n + \vec{e}_d} f) (D^{\vec{\beta}_n^*} g). \end{aligned}$$

□

Theorem 3.5. When the star product with separation of variables for smooth functions f and g on a locally symmetric Kähler manifold is given as

$$f * g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n (D^{\vec{\alpha}_n} f) (D^{\vec{\beta}_n^*} g),$$

these covariantly constants $T_{\vec{\alpha}_n \vec{\beta}_n^*}^n$ are determined by the following recurrence relations

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \beta_i T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^* - \vec{e}_i} g \right) - \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} g_{id} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n + \vec{e}_d} f \right) \left(D^{\vec{\beta}_n^*} g \right) \\
& + \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \sum_{k=1}^N \sum_{p=1}^N \frac{\beta_k^n (\beta_k^n - 1)}{2} R_{\vec{p}}^{\bar{k} \bar{k}} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^* + \vec{e}_p - 2\vec{e}_k} g \right) \\
& + \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} \sum_{\rho=1}^N \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \beta_k^n \beta_{k+l}^n R_{\vec{\rho}}^{\overline{k+l} \bar{k}} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^* + \vec{e}_{\rho} - \vec{e}_k - \vec{e}_{k+l}} g \right) \\
& = 0
\end{aligned}$$

Proof. $0 = [L_f, \partial_i \Phi + \hbar \partial_i] g$ is the condition that determines the star product. $[L_f, \partial_i \Phi] g$ and $[L_f, \hbar \partial_i] g$ were calculated in Proposition 3.2 and 3.4. \square

Theorem 3.6. When the star product with separation of variables for smooth functions f and g on a local symmetric Kähler manifold is given as

$$f * g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n \vec{\beta}_n^*} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^*} g \right),$$

these smooth functions $T_{\vec{\alpha}_n \vec{\beta}_n^*}^n$, which are covariantly constants, are determined by the following recurrence relations for $\forall i$:

$$\begin{aligned}
& \sum_{d=1}^N \hbar g_{id} T_{\vec{\alpha}_n - \vec{e}_d \vec{\beta}_n^* - \vec{e}_i}^{n-1} \\
& = \beta_i T_{\vec{\alpha}_n \vec{\beta}_n^*}^n + \sum_{k=1}^N \sum_{p=1}^N \frac{\hbar (\beta_k^n - \delta_{kp} - \delta_{ik} + 1) (\beta_k^n - \delta_{kp} - \delta_{ik} + 2)}{2} R_{\vec{p}}^{\bar{k} \bar{k}} T_{\vec{\alpha}_n \vec{\beta}_n^* - \vec{e}_p + 2\vec{e}_k - \vec{e}_i}^n \\
& + \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{p=1}^N \hbar (\beta_k^n - \delta_{kp} - \delta_{ik} + 1) (\beta_{k+l}^n - \delta_{(k+l)p} - \delta_{i,(k+l)} + 1) R_{\vec{p}}^{\overline{k+l} \bar{k}} T_{\vec{\alpha}_n \vec{\beta}_n^* - \vec{e}_p + \vec{e}_k + \vec{e}_{k+l} - \vec{e}_i}^n.
\end{aligned}$$

Proof. Changing the summation of Theorem 3.5,

$$\begin{aligned}
& \hbar \sum_{n=1}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} \sum_{d=1}^N g_{id} T_{\vec{\alpha}_n - \vec{e}_d, \vec{\beta}_n^* - \vec{e}_i}^{n-1} \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^* - \vec{e}_i} g \right) \\
&= \sum_{n=0}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} \beta_i T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^* - \vec{e}_i} g \right) \\
&\quad + \hbar \sum_{n=0}^{\infty} \sum_{k=1}^N \sum_{p=1}^N \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} \frac{(\beta_k^n - \delta_{kp} - \delta_{ik} + 1)(\beta_k^n - \delta_{kp} - \delta_{ik} + 2)}{2} R_{\vec{p}}^{\vec{k}\vec{k}}_{\vec{i}} \\
&\quad \times T_{\vec{\alpha}_n, \vec{\beta}_n^* - \vec{e}_p + 2\vec{e}_k - \vec{e}_i}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^* - \vec{e}_i} g \right) \\
&\quad + \hbar \sum_{n=0}^{\infty} \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{p=1}^N \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} (\beta_k^n - \delta_{kp} - \delta_{ik} + 1) (\beta_{k+l}^n - \delta_{(k+l)p} - \delta_{i,(k+l)} + 1) R_{\vec{p}}^{\vec{k}+\vec{l}\vec{k}}_{\vec{i}} \\
&\quad \times T_{\vec{\alpha}_n, \vec{\beta}_n^* - \vec{e}_p + \vec{e}_k + \vec{e}_{k+l} - \vec{e}_i}^n \left(D^{\vec{\alpha}_n} f \right) \left(D^{\vec{\beta}_n^* - \vec{e}_i} g \right),
\end{aligned}$$

and this implies the theorem. \square

4 One and two dimensional cases

By using Theorem 3.6 we can provide explicit star products for locally symmetric Kähler manifolds. In this section, an explicit expression of a star product of a one-dimensional locally symmetric Kähler manifold is constructed as an example. A two-dimensional locally symmetric Kähler manifold is also considered.

At first, we study an explicit expression of a star product of a one-dimensional locally symmetric Kähler manifold. A formal discussions are given in [32], and star products are studied in [25]. Complex surfaces with arbitrary genus are known as a example of such manifolds when we chose proper coordinates and metrics. The Scalar curvature R is defined as

$$R = g^{i\bar{j}} R_{i\bar{j}} = R_{\bar{i}}^{\bar{j}l} \bar{l}_{\bar{j}}.$$

Proposition 4.1. *Let M be a one-dimensional locally symmetric Kähler manifold ($N = 1$) and f and g be smooth functions on M . The star product with separation of variables for f and g can be described as*

$$f * g = \sum_{n=0}^{\infty} \left[\left(g^{1\bar{1}} \right)^n \left\{ \prod_{k=1}^{n-1} \frac{2\hbar}{2k + \hbar k(k-1)R} \right\} \left\{ \left(g^{1\bar{1}} \frac{\partial}{\partial z} \right)^n f \right\} \left\{ \left(g^{1\bar{1}} \frac{\partial}{\partial \bar{z}} \right)^n g \right\} \right]$$

where

$$R = R_1^{\bar{1}\bar{1}}_{\bar{1}}.$$

Proof. $N = 1, i = 1$ and

$$D^{\alpha_n} f = \left(g^{1\bar{1}} \frac{\partial}{\partial z} \right)^n f, \quad D^{\beta_n^*} g = \left(g^{1\bar{1}} \frac{\partial}{\partial \bar{z}} \right)^n g$$

are substituted in Theorem 3.5, then we obtain

$$\hbar \sum_{n=1}^{\infty} g^{1\bar{1}} T^{n-1} (D^n f_1) (D^{n-1} f_2) = \sum_{n=0}^{\infty} \left\{ n + \frac{\hbar n (n-1)}{2} R_1^{\bar{1}\bar{1}}_{\bar{1}} \right\} T^n (D^n f_1) (D^{n-1} f_2)$$

or equivalently, the recurrence relation of T^n is given as

$$T^n = g^{1\bar{1}} \left\{ \frac{2\hbar}{2n + \hbar n (n-1) R} \right\} T^{n-1}.$$

From Proposition 3.1 the first term T^1 is given as $T^1 = \hbar g^{1\bar{1}}$. Then, T^n is given as

$$T^n = \left(g^{1\bar{1}} \right)^n \prod_{k=1}^{n-1} \left\{ \frac{2\hbar}{2k + \hbar k (k-1) R} \right\}.$$

□

Next, we discuss star products on general two-dimensional locally symmetric Kähler manifolds.

According to Proposition 3.1, for a two-dimensional locally symmetric Kähler manifold M , $T^1_{\alpha_1 \beta_1^*}$ is given as

$$\begin{pmatrix} T^1_{(1,0),(1,0)} & T^1_{(1,0),(0,1)} \\ T^1_{(0,1),(1,0)} & T^1_{(0,1),(0,1)} \end{pmatrix} = \hbar \begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix}.$$

Next, we estimate $T^2_{\alpha_2 \beta_2^*}$.

Proposition 4.2. Let M be a two-dimensional locally symmetric Kähler manifold and f and g be smooth functions on M . $T^2_{\alpha_2 \beta_2^*}$ given in (3.3) is obtained by

$$\begin{pmatrix} T^2_{(2,0),(2,0)} & T^2_{(2,0),(1,1)} & T^2_{(2,0),(0,2)} \\ T^2_{(1,1),(2,0)} & T^2_{(1,1),(1,1)} & T^2_{(1,1),(0,2)} \\ T^2_{(0,2),(2,0)} & T^2_{(0,2),(1,1)} & T^2_{(0,2),(0,2)} \end{pmatrix} \\ = \hbar^2 \begin{pmatrix} (g_{1\bar{1}})^2 & g_{1\bar{1}} g_{2\bar{1}} & (g_{2\bar{1}})^2 \\ 2g_{1\bar{1}} g_{1\bar{2}} & g_{2\bar{1}} g_{1\bar{2}} + g_{1\bar{1}} g_{2\bar{2}} & 2g_{2\bar{1}} g_{2\bar{2}} \\ (g_{1\bar{2}})^2 & g_{2\bar{1}} g_{2\bar{2}} & (g_{2\bar{2}})^2 \end{pmatrix} \begin{pmatrix} 2 + \hbar R_1^{\bar{1}\bar{1}}_{\bar{1}} & \hbar R_2^{\bar{1}\bar{1}}_{\bar{1}} & \hbar R_2^{\bar{1}\bar{1}}_{\bar{2}} \\ \hbar R_1^{\bar{2}\bar{1}}_{\bar{1}} & 1 + \hbar R_2^{\bar{2}\bar{1}}_{\bar{1}} & \hbar R_2^{\bar{2}\bar{1}}_{\bar{2}} \\ \hbar R_1^{\bar{2}\bar{2}}_{\bar{1}} & \hbar R_2^{\bar{2}\bar{2}}_{\bar{1}} & 2 + \hbar R_2^{\bar{2}\bar{2}}_{\bar{2}} \end{pmatrix}^{-1}.$$

The proof is given in appendix A.

5 Deformation quantization for complex Grassmann manifold

In this section, recurrence relations to obtain star products on complex Grassmann manifolds are derived. Especially we calculate star products of $\mathbb{C}P^N$. Note that this star product is also equal to the ones given in [5, 12, 28], and if we put a some restriction our star product is also equal to the one given in [1], as they are shown in [27, 21]. The equivalence is also discussed in [31, 32]. In addition, recurrence relations to construct star products for $G_{2,2}$ was derived. Deformation quantization of Grassmann manifolds and flag manifolds were studied in [17, 10, 11, 24].

Complex Grassmann manifold $G_{p,q}$ is defined as a set of the whole p dimensional part vector space of $p+q$ dimensional vector space V . The local coordinate can be defined in a similar way to S. Kobayashi and K. Nomizu pp.160-162[19].

Let U be an open subset of $G_{p,q}$. A chart (U, ϕ) is defined by

$$U := \left\{ Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \in M(p+q, p; \mathbb{C}) ; |Y_0| \neq 0 \right\}$$

and

$$\phi : U \longrightarrow M(q, p; \mathbb{C})$$

where

$$\phi(Y) = Y_1 Y_0^{-1}.$$

This is a holomorphic map of U onto an open subset of $p \times q$ -dimensional complex space.

In this section, capital letter indices $A, B, C \dots$ mean $aa', bb', cc' \dots$. In the inhomogeneous coordinates $z^I := z^{ii'}$, $z^{\bar{I}} := z^{\bar{i}\bar{i}'}$, ($i = 1, 2, \dots, p, i' = 1, 2, \dots, q$), the Kähler potential of $G_{p,q}$ is given as

$$\Phi = \ln \left| E_q + Z^\dagger Z \right|, \quad (5.1)$$

where $Z = \phi(Y) = (z^I) \in M(q, p; \mathbb{C})$ and $E_q \in M(q, q; \mathbb{C})$ is the unite matrix. From (5.1), the following facts are derived.

Fact 5.1. *The Fubini-Study metric $(g_{I\bar{J}})$ is*

$$ds^2 = 2g_{I\bar{J}} dz^I d\bar{z}^{\bar{J}},$$

where

$$g_{I\bar{J}} := g_{ii'\bar{j}\bar{j}'} = \partial_I \partial_{\bar{J}} \Phi = a^{j\bar{i}} b^{i'j'}, \quad g^{I\bar{J}} := g^{ii'\bar{j}\bar{j}'} = a_{ij} b_{j'i'}.$$

with

$$a_{ij} = \delta_{ij} + z^{ik'} \bar{z}^{j\bar{k}'}, \quad b_{i'j'} = \delta_{i'j'} + \bar{z}^{ki'} z^{kj'}.$$

Fact 5.2. *The curvature of a complex Grassmann manifold is*

$$R_{\bar{A}}^{\bar{C}\bar{D}}{}_{\bar{B}} = g^{P\bar{C}} g^{Q\bar{D}} R_{\bar{A}PQ\bar{B}} = -\delta_{\bar{a}b'}^{\bar{c}} \delta_{\bar{b}a'}^{\bar{d}} \bar{D} - \delta_{\bar{b}a'}^{\bar{c}} \delta_{\bar{a}b'}^{\bar{d}} \bar{D}, \quad (5.2)$$

where

$$\delta_{\bar{a}b'}^{\bar{c}} = \begin{cases} 1 & (a = c, b' = d') \\ 0 & (\text{otherwise}) \end{cases}.$$

From these facts, we can derive the recurrence relations to determine star products on the Grassmann manifolds.

5.1 Some preparations

A function similar to the determinant is defined on the matrix space.

Definition 4. Let $C = (C_{k,l})_{1 \leq k \leq n, 1 \leq l \leq n}$ be a $n \times n$ matrix. We define $|\cdot|^+$ as a \mathbb{C} -valued function on $M(n, n; \mathbb{C})$ such that

$$|C|^+ := \sum_{\sigma_n \in S_n} \prod_{k=1}^n C_{k, \sigma_n(k)}.$$

Example 1. *Here we show some examples. These suggest some properties like determinant.*

1.

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^+ = c_{11}c_{22} + c_{12}c_{21}$$

2.

$$\begin{aligned} & \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}^+ \\ &= c_{11}c_{22}c_{33} + c_{11}c_{23}c_{32} + c_{12}c_{21}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} + c_{13}c_{22}c_{31} \\ &= c_{11} \begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix}^+ + c_{12} \begin{vmatrix} c_{11} & c_{13} \\ c_{31} & c_{33} \end{vmatrix}^+ + c_{13} \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^+ \end{aligned}$$

Remark. Similar to a determinant

$$|^t C|^+ = |C|^+,$$

where $^t C$ is a transposed matrix of C .

The following is a proposition similar to cofactor expansion of a determinant.

Proposition 5.3.

$$|C|^+ = \left| \begin{array}{ccccc} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nn} \end{array} \right|^+ = \sum_{j=1}^n c_{ij} \left| \begin{array}{ccccc} c_{11} & \cdots & \hat{c}_{1j} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{c}_{i1} & \cdots & \hat{c}_{ij} & \cdots & \hat{c}_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & \hat{c}_{nj} & \cdots & c_{nn} \end{array} \right|^+$$

Proof. A proof for this function is similar to the case of determinants. \square

Definition 5. A matrix $G^{\vec{\alpha}_n, \vec{\beta}_n^*}$ is defined by using the Riemannian metrics on M . Its elements are metrics on M and are located as follows. $\vec{\alpha}_n$ and $\vec{\beta}_n$ are elements of \mathbb{Z}^N .

$$G^{\vec{\alpha}_n, \vec{\beta}_n^*} = \begin{pmatrix} \tilde{G}_{11} & \cdots & \tilde{G}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{G}_{n1} & \cdots & \tilde{G}_{nn} \end{pmatrix}$$

where

$$\tilde{G}_{pq} =: g_{p\bar{q}} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in M(\alpha_p^n, \beta_q^n; \mathbb{C})$$

i.e.

$$G^{\vec{\alpha}_n, \vec{\beta}_n^*} = \left(\underbrace{\begin{pmatrix} g_{1\bar{1}} & \cdots & g_{1\bar{1}} \\ \vdots & \ddots & \vdots \\ g_{1\bar{1}} & \cdots & g_{1\bar{1}} \end{pmatrix}}_{\beta_1^n} \cdots \underbrace{\begin{pmatrix} g_{1\bar{N}} & \cdots & g_{1\bar{N}} \\ \vdots & \ddots & \vdots \\ g_{1\bar{N}} & \cdots & g_{1\bar{N}} \end{pmatrix}}_{\beta_N^n} \right) \left. \begin{matrix} \left. \begin{matrix} \left. \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \right\} \alpha_1^n \\ \vdots \end{matrix} \right\} \cdot \\ \left. \begin{matrix} \left. \begin{matrix} g_{N\bar{1}} & \cdots & g_{N\bar{1}} \\ \vdots & \ddots & \vdots \\ g_{N\bar{1}} & \cdots & g_{N\bar{1}} \end{matrix} \right\} \alpha_N^n \end{matrix} \right\} \end{matrix} \right\}$$

For example $N = 2, \vec{\alpha}_3 = (2, 1), \vec{\beta}_3^* = (1, 2)^*$, then $G^{\vec{\alpha}_3, \vec{\beta}_3^*}$ is determined as

$$G^{\vec{\alpha}_3, \vec{\beta}_3^*} = \left(\begin{array}{c|cc} g_{1\bar{1}} & g_{1\bar{2}} & g_{1\bar{2}} \\ g_{1\bar{1}} & g_{1\bar{2}} & g_{1\bar{2}} \\ \hline g_{2\bar{1}} & g_{2\bar{2}} & g_{2\bar{2}} \end{array} \right).$$

From Proposition 5.3, we obtain the following corollary.

Corollary 5.4. For a matrix $G^{\vec{\alpha}_n, \vec{\beta}_n^*}$,

$$\left| G^{\vec{\alpha}_n, \vec{\beta}_n^*} \right|^+ = \sum_{J=1}^N \beta_J^n g_{JI} \left| G^{\vec{\alpha}_n - \vec{e}_I, \vec{\beta}_n^* - \vec{e}_J} \right|^+ = \sum_{K=1}^N \alpha_K^n g_{IK} \left| G^{\vec{\alpha}_n - \vec{e}_K, \vec{\beta}_n^* - \vec{e}_I} \right|^+.$$

5.2 Deformation quantization for a complex projective space

In this subsection, we obtain concrete expression of star products on $\mathbb{C}P^N$. A complex projective space $\mathbb{C}P^N$ is a Grassmann manifold $G_{1,N}$ by definition.

Proposition 5.5. Let M be a complex projective space and f and g be smooth functions on M . The recurrence relation of $T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n$ given in (3.3) is

$$T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n = \sum_{d=1}^N \frac{\hbar g_{id}}{(1 + \hbar - \hbar n)} T_{\vec{\alpha}_n - \vec{e}_d, \vec{\beta}_n^* - \vec{e}_i}^{n-1}. \quad (5.3)$$

Proof. The curvature (5.2) is substituted for Theorem 3.6, and the following is proved.

$$\begin{aligned} & \sum_{d=1}^N \hbar g_{id} T_{\vec{\alpha}_n - \vec{e}_d, \vec{\beta}_n^* - \vec{e}_i}^{n-1} \\ &= \beta_i T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n + \sum_{k=1}^N \sum_{p=1}^N \frac{\hbar (\beta_k^n - \delta_{kp} - \delta_{ik} + 1) (\beta_k^n - \delta_{kp} - \delta_{ik} + 2)}{2} R_{\vec{p}}^{\vec{k}\vec{k}}{}_{\vec{i}} T_{\vec{\alpha}_n, \vec{\beta}_n^* - \vec{e}_p + 2\vec{e}_k - \vec{e}_i}^n \\ & \quad + \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{p=1}^N \hbar (\beta_k^n - \delta_{kp} - \delta_{ik} + 1) (\beta_{k+l}^n - \delta_{(k+l)p} - \delta_{i, (k+l)} + 1) R_{\vec{p}}^{\vec{k}+\vec{l}\vec{k}}{}_{\vec{i}} T_{\vec{\alpha}_n, \vec{\beta}_n^* - \vec{e}_p + \vec{e}_k + \vec{e}_{k+l} - \vec{e}_i}^n. \end{aligned}$$

We also use

$$R_{\vec{p}}^{\vec{k}+\vec{l}\vec{k}}{}_{\vec{i}} = -\delta_{\vec{p}, \vec{k}+\vec{l}} \delta_{\vec{i}, \vec{k}} - \delta_{\vec{i}, \vec{k}+\vec{l}} \delta_{\vec{p}, \vec{k}}, \quad R_{\vec{p}}^{\vec{k}\vec{k}}{}_{\vec{i}} = -\delta_{\vec{p}, \vec{k}} \delta_{\vec{i}, \vec{k}} - \delta_{\vec{i}, \vec{k}} \delta_{\vec{p}, \vec{k}},$$

then the above is rewritten as

$$\begin{aligned} & \sum_{d=1}^N \hbar g_{id} T_{\vec{\alpha}_n - \vec{e}_d, \vec{\beta}_n^* - \vec{e}_i}^{n-1} \\ &= \beta_i T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n - \sum_{k=1}^N \sum_{p=1}^N \frac{\hbar (\beta_k^n - \delta_{kp} - \delta_{ik} + 1) (\beta_k^n - \delta_{kp} - \delta_{ik} + 2) (\delta_{\vec{p}, \vec{k}} \delta_{\vec{i}, \vec{k}} + \delta_{\vec{i}, \vec{k}} \delta_{\vec{p}, \vec{k}})}{2} T_{\vec{\alpha}_n, \vec{\beta}_n^* - \vec{e}_p + 2\vec{e}_k - \vec{e}_i}^n \\ & \quad - \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{p=1}^N \hbar (\beta_k^n - \delta_{kp} - \delta_{ik} + 1) (\beta_{k+l}^n - \delta_{(k+l)p} - \delta_{i, (k+l)} + 1) \\ & \quad \times (\delta_{\vec{p}, \vec{k}+\vec{l}} \delta_{\vec{i}, \vec{k}} + \delta_{\vec{i}, \vec{k}+\vec{l}} \delta_{\vec{p}, \vec{k}}) T_{\vec{\alpha}_n, \vec{\beta}_n^* - \vec{e}_p + \vec{e}_k + \vec{e}_{k+l} - \vec{e}_i}^n. \end{aligned}$$

The theorem follows from this. \square

Theorem 5.6. *Let f and g be smooth functions on a projective space \mathbb{CP}^N . A star product with separation of variables on a projective space \mathbb{CP}^N is given as*

$$f * g = f \cdot g + \sum_{n=1}^{\infty} \sum_{\vec{\alpha}_n, \vec{\beta}_n^*} \left| G^{\vec{\alpha}_n, \vec{\beta}_n^*} \right|^+ \left\{ \prod_{k=0}^n \frac{\hbar}{(1 + \hbar - \hbar k) \alpha_k^n! \beta_k^n!} \right\} (D^{\vec{\alpha}_n} f) (D^{\vec{\beta}_n^*} g). \quad (5.4)$$

Proof. *We show that*

$$T_{\vec{\alpha}_n, \vec{\beta}_n^*}^n = \left| G^{\vec{\alpha}_n, \vec{\beta}_n^*} \right|^+ \left\{ \prod_{k=0}^n \frac{\hbar}{(1 + \hbar - \hbar k) \alpha_k^n! \beta_k^n!} \right\}$$

satisfies (5.3). The R.H.S of (5.3) for this case is given as

$$\sum_{d=1}^N \frac{\hbar g_{id}}{(1 + \hbar - \hbar n) \beta_i^n} T_{\vec{\alpha}_n - \vec{e}_d, \vec{\beta}_n^* - \vec{e}_i}^{n-1} = \sum_{d=1}^N g_{id} \alpha_d^n \left| G^{\vec{\alpha}_n - \vec{e}_d, \vec{\beta}_n^* - \vec{e}_i} \right|^+ \frac{\hbar}{(1 + \hbar - \hbar n)} \prod_{k=0}^{n-1} \frac{\hbar}{(1 + \hbar - \hbar k) \alpha_k^n! \beta_k^n!}.$$

Using Corollary 5.4, R.H.S. of the above is rewritten as

$$\left| G^{\vec{\alpha}_n, \vec{\beta}_n^*} \right|^+ \prod_{k=0}^n \frac{\hbar}{(1 + \hbar - \hbar k) \alpha_k^n! \beta_k^n!}.$$

This shows the given $T_{\vec{\alpha}_n, \vec{\beta}_n^}^n$ satisfies the recurrence relation (5.3). \square*

Fact 5.7. *Let f and g be smooth functions on a projective space \mathbb{CP}^N . A star product on a projective space \mathbb{CP}^N is given in [28] as*

$$\begin{aligned} f \tilde{*} g &= \sum_{n=0}^{\infty} \frac{\Gamma(1 - n + 1/\hbar) g^{\bar{j}_1 k_1} \dots g^{\bar{j}_n k_n}}{n! \Gamma(1 + 1/\hbar)} (\nabla_{\bar{j}_1} \dots \nabla_{\bar{j}_n} f) (\nabla_{k_1} \dots \nabla_{k_n} g) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(1 - n + 1/\hbar)}{n! \Gamma(1 + 1/\hbar)} (D^{k_1} \dots D^{k_n} f) (\nabla_{k_1} \dots \nabla_{k_n} g) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(1 - n + 1/\hbar) g_{\bar{m}_1 k_1} \dots g_{\bar{m}_n k_n}}{n! \Gamma(1 + 1/\hbar)} (D^{k_1} \dots D^{k_n} f) (D^{\bar{m}_1} \dots D^{\bar{m}_n} g). \end{aligned} \quad (5.5)$$

As mentioned in Section 2, the star product with separation of variables is uniquely determined. This fact means (5.4) coincides with (5.5). This coincidence is easily checked from Definition 5.

5.3 Deformation quantization for a $G_{2,2}$

In this subsection, we derive the recurrence relation to obtain concrete expression of star products on a Grassmann manifold $G_{2,2}$. The inhomogeneous coordinates are $z^{11'}, z^{12'}, z^{21'}$ and $z^{22'}$. To decide the order of coordinates is useful in order to calculate the finite sum. We set the order: $11' < 12' < 21' < 22'$. In this subsection, j is used as “Not i ”. That means that if $i = 1$ then $j = 2$ and if $i = 2$ then $j = 1$. For example, if $I = ii' = 11'$, then $ij' = 12', ji' = 21', J = 22'$. If $I = ii' = 12'$, then $ij' = 11', ji' = 22', J = 21'$. A finite sum is defined as

$$\sum_{D=1}^4 a_D := a_{11'} + a_{12'} + a_{21'} + a_{22'}.$$

Theorem 5.8. *Let f and g be smooth functions on $G_{2,2}$. The recurrence relation of $T_{\vec{\alpha}_n \vec{\beta}_n^*}^n$ given in (3.3) is*

$$\begin{aligned} & \beta_I (1 + \hbar - \hbar \beta_I^n - \hbar \beta_{ji'}^n - \hbar \beta_{ij'}^n) T_{\vec{\alpha}_n \vec{\beta}_n^*}^n - \hbar (\beta_{ij'}^n + 1) (\beta_{ji'}^n + 1) T_{\vec{\alpha}_n \vec{\beta}_n^* - \vec{e}_J + \vec{e}_{ij'} + \vec{e}_{ji'} - \vec{e}_I}^n \\ &= \hbar g_{II} T_{\vec{\alpha}_n - \vec{e}_I \vec{\beta}_n^* - \vec{e}_I}^{n-1} + \hbar g_{Iij'} T_{\vec{\alpha}_n - \vec{e}_{ij'} \vec{\beta}_n^* - \vec{e}_I}^{n-1} + \hbar g_{Iji'} T_{\vec{\alpha}_n - \vec{e}_{ji'} \vec{\beta}_n^* - \vec{e}_I}^{n-1} + \hbar g_{IJ} T_{\vec{\alpha}_n - \vec{e}_J \vec{\beta}_n^* - \vec{e}_I}^{n-1}. \end{aligned} \quad (5.6)$$

for each I .

Proof. The curvature (5.2) is substituted into Theorem 3.6, and the following is obtained.

$$\begin{aligned} & \sum_{D=1}^4 \hbar g_{ID} T_{\vec{\alpha}_n - \vec{e}_D \vec{\beta}_n^* - \vec{e}_I}^{n-1} \\ &= \beta_I T_{\vec{\alpha}_n \vec{\beta}_n^*}^n \\ & - \sum_{K=1}^4 \sum_{P=1}^4 \frac{\hbar (\beta_K^n - \delta_{KP} - \delta_{IK} + 1) (\beta_K^n - \delta_{KP} - \delta_{IK} + 2) \left(\delta_{\vec{p}i', \vec{K}} \delta_{\vec{i}p', \vec{K}} + \delta_{\vec{i}p', \vec{K}} \delta_{\vec{p}i', \vec{K}} \right)}{2} T_{\vec{\alpha}_n \vec{\beta}_n^* - \vec{e}_P + 2\vec{e}_K - \vec{e}_I}^n \\ & - \sum_{K=1}^{4-1} \sum_{L=1}^{4-K} \sum_{P=1}^4 \hbar (\beta_K^n - \delta_{KP} - \delta_{IK} + 1) (\beta_{K+L}^n - \delta_{(K+L), P} - \delta_{I, (K+L)} + 1) \\ & \quad \times \left(\delta_{\vec{p}i', \vec{K}+\vec{L}} \delta_{\vec{i}p', \vec{K}} + \delta_{\vec{i}p', \vec{K}+\vec{L}} \delta_{\vec{p}i', \vec{K}} \right) T_{\vec{\alpha}_n \vec{\beta}_n^* - \vec{e}_P + \vec{e}_K + \vec{e}_{K+L} - \vec{e}_I}^n \\ &= \beta_I \{ 1 + \hbar - \hbar \beta_I^n - \hbar \beta_{ji'}^n - \hbar \beta_{ij'}^n \} T_{\vec{\alpha}_n \vec{\beta}_n^*}^n - \hbar (\beta_{ij'}^n + 1) (\beta_{ji'}^n + 1) T_{\vec{\alpha}_n \vec{\beta}_n^* - \vec{e}_J + \vec{e}_{ij'} + \vec{e}_{ji'} - \vec{e}_I}^n \end{aligned}$$

The theorem follows from this. \square

Star products on a noncommutative $G_{2,2}$ are determined by this formula recursively. For general $G_{p,q}$, the recurrence relations are determined in a similar way.

6 Conclusion

In this article, noncommutative locally symmetric Kähler manifolds have been studied by using deformation quantization given by Karabegov [15], which is called deformation quantization with separation of variables. The similar approach was already tried in [27], and recursion relations and explicit expression of star products for $\mathbb{C}P^N$ and $\mathbb{C}H^N$ are also given in it. In this article, we improve the method in [27] to more useful one. The point to obtain the concrete star products is to translate the system of PDE into algebraic recursion relations. The fact that the Riemannian curvature tensor is given as a covariantly constant removes the complication of the system of PDE. From the results, we can construct explicit star products order by order by using only algebraic calculations. As an example, a concrete expression of a star product for $\dim_{\mathbb{C}} M = 1$ locally symmetric Kähler manifolds was obtained in Section 4. It is known that Riemann surfaces with arbitrary genus are possible to be described as such locally symmetric Kähler manifolds under proper settings. For $\dim_{\mathbb{C}} M = 2$ case, a star product was explicitly given until the second order of derivative, too. The Grassmann manifolds are typically manifolds of symmetric Kähler manifolds. In Section 5, we also studied the algebraic recursion relations for star products for the Grassmann manifolds, too. As an example, we constructed a star product for $\mathbb{C}P^N$, that is equal to the star product in [27], but it has a different expression.

Before the end of this article, we make two comments. The first one is about the representation of the noncommutative manifolds given by the deformation quantization with separation of variables. In [29], the Fock representation for noncommutative $\mathbb{C}P^N$ is constructed. Using new star products with separation of variables for Riemann surfaces given in this article, we can make such kind of Fock representation, similarly. Indeed, the recipe to construct the twisted Fock representation is already constructed for general Kähler manifolds in [30]. The second comment is about the star products for general Grassmann manifolds. In this article, star products on $\mathbb{C}P^N$ are given as an easy case of Grassmann manifolds. It is expected that explicit star products for the general Grassmann manifolds are obtained by the similar way of this article. This problem is left for a future work.

Acknowledgments

A.S. was supported in part by JSPS KAKENHI Grant Number 16K05138.

A Proof for Proposition 4.2

$N = 2$ and $n = 2$ are substituted in Theorem 3.6. The results are listed here. $\vec{\alpha}_2, \vec{\beta}_2^* \in \{(2, 0), (1, 1), (0, 2)\}$ and $i = \{1, 2\}$.

$$\begin{aligned}
\hbar^2 (g_{\bar{1}1})^2 &= \left(2 + \hbar R_{\bar{1}}^{\bar{1}\bar{1}}\right) T_{(2,0),(2,0)}^2 + \hbar R_{\bar{1}}^{\bar{2}\bar{2}} T_{(2,0),(0,2)}^2 + \hbar R_{\bar{1}}^{\bar{2}\bar{1}} T_{(2,0),(1,1)}^2 \\
\hbar^2 (g_{\bar{1}2})^2 &= \left(2 + \hbar R_{\bar{1}}^{\bar{1}\bar{1}}\right) T_{(0,2),(2,0)}^2 + \hbar R_{\bar{1}}^{\bar{2}\bar{2}} T_{(0,2),(0,2)}^2 + \hbar R_{\bar{1}}^{\bar{2}\bar{1}} T_{(0,2),(1,1)}^2 \\
2\hbar^2 g_{\bar{1}1} g_{\bar{1}2} &= \left(2 + \hbar R_{\bar{1}}^{\bar{1}\bar{1}}\right) T_{(1,1),(2,0)}^2 + \hbar R_{\bar{1}}^{\bar{2}\bar{2}} T_{(1,1),(0,2)}^2 + \hbar R_{\bar{1}}^{\bar{2}\bar{1}} T_{(1,1),(1,1)}^2 \\
\hbar^2 g_{\bar{1}1} g_{\bar{2}1} &= \left(1 + \hbar R_{\bar{2}}^{\bar{2}\bar{1}}\right) T_{(2,0),(1,1)}^2 + \hbar R_{\bar{2}}^{\bar{1}\bar{1}} T_{(2,0),(2,0)}^2 + \hbar R_{\bar{2}}^{\bar{2}\bar{2}} T_{(2,0),(0,2)}^2 \\
\hbar^2 g_{\bar{1}2} g_{\bar{2}2} &= \left(1 + \hbar R_{\bar{2}}^{\bar{2}\bar{1}}\right) T_{(0,2),(1,1)}^2 + \hbar R_{\bar{2}}^{\bar{1}\bar{1}} T_{(0,2),(2,0)}^2 + \hbar R_{\bar{2}}^{\bar{2}\bar{2}} T_{(0,2),(0,2)}^2 \\
\hbar^2 g_{\bar{1}1} g_{\bar{2}2} + \hbar^2 g_{\bar{2}1} g_{\bar{1}2} &= \left(1 + \hbar R_{\bar{2}}^{\bar{2}\bar{1}}\right) T_{(1,1),(1,1)}^2 + \hbar R_{\bar{2}}^{\bar{1}\bar{1}} T_{(1,1),(2,0)}^2 + \hbar R_{\bar{2}}^{\bar{2}\bar{2}} T_{(1,1),(0,2)}^2 \\
\hbar^2 (g_{\bar{2}1})^2 &= \left(2 + \hbar R_{\bar{2}}^{\bar{2}\bar{2}}\right) T_{(2,0),(0,2)}^2 + \hbar R_{\bar{2}}^{\bar{1}\bar{1}} T_{(2,0),(2,0)}^2 + \hbar R_{\bar{2}}^{\bar{2}\bar{1}} T_{(2,0),(1,1)}^2 \\
\hbar^2 (g_{\bar{2}2})^2 &= \left(2 + \hbar R_{\bar{2}}^{\bar{2}\bar{2}}\right) T_{(0,2),(0,2)}^2 + \hbar R_{\bar{2}}^{\bar{1}\bar{1}} T_{(0,2),(2,0)}^2 + \hbar R_{\bar{2}}^{\bar{2}\bar{1}} T_{(0,2),(1,1)}^2 \\
2\hbar^2 g_{\bar{2}1} g_{\bar{2}2} &= \left(2 + \hbar R_{\bar{2}}^{\bar{2}\bar{2}}\right) T_{(1,1),(0,2)}^2 + \hbar R_{\bar{2}}^{\bar{1}\bar{1}} T_{(1,1),(2,0)}^2 + \hbar R_{\bar{2}}^{\bar{2}\bar{1}} T_{(1,1),(1,1)}^2 \\
\hbar^2 g_{\bar{1}1} g_{\bar{2}1} &= \left(1 + \hbar R_{\bar{1}}^{\bar{2}\bar{1}}\right) T_{(2,0),(1,1)}^2 + \hbar R_{\bar{1}}^{\bar{1}\bar{1}} T_{(2,0),(2,0)}^2 + \hbar R_{\bar{1}}^{\bar{2}\bar{2}} T_{(2,0),(0,2)}^2 \\
\hbar^2 g_{\bar{2}1} g_{\bar{2}2} &= \left(1 + \hbar R_{\bar{1}}^{\bar{2}\bar{1}}\right) T_{(0,2),(1,1)}^2 + \hbar R_{\bar{1}}^{\bar{1}\bar{1}} T_{(0,2),(2,0)}^2 + \hbar R_{\bar{1}}^{\bar{2}\bar{2}} T_{(0,2),(0,2)}^2 \\
\hbar^2 g_{\bar{2}1} g_{\bar{1}2} + \hbar^2 g_{\bar{1}1} g_{\bar{2}2} &= \left(1 + \hbar R_{\bar{1}}^{\bar{2}\bar{1}}\right) T_{(1,1),(1,1)}^2 + \hbar R_{\bar{1}}^{\bar{1}\bar{1}} T_{(1,1),(2,0)}^2 + \hbar R_{\bar{1}}^{\bar{2}\bar{2}} T_{(1,1),(0,2)}^2.
\end{aligned}$$

There are multiple overlapping and tautological equations are omitted.

With (3.1) these are the same as the following equation.

$$\begin{aligned}
&\hbar^2 \begin{pmatrix} (g_{\bar{1}1})^2 & g_{\bar{1}1} g_{\bar{2}1} & (g_{\bar{2}1})^2 \\ 2g_{\bar{1}1} g_{\bar{1}2} & g_{\bar{2}1} g_{\bar{1}2} + g_{\bar{1}1} g_{\bar{2}2} & 2g_{\bar{2}1} g_{\bar{2}2} \\ (g_{\bar{1}2})^2 & g_{\bar{2}1} g_{\bar{2}2} & (g_{\bar{2}2})^2 \end{pmatrix} \\
&= \begin{pmatrix} T_{(2,0),(2,0)}^2 & T_{(2,0),(1,1)}^2 & T_{(2,0),(0,2)}^2 \\ T_{(1,1),(2,0)}^2 & T_{(1,1),(1,1)}^2 & T_{(1,1),(0,2)}^2 \\ T_{(0,2),(2,0)}^2 & T_{(0,2),(1,1)}^2 & T_{(0,2),(0,2)}^2 \end{pmatrix} \begin{pmatrix} 2 + \hbar R_{\bar{1}}^{\bar{1}\bar{1}} & \hbar R_{\bar{2}}^{\bar{1}\bar{1}} & \hbar R_{\bar{2}}^{\bar{1}\bar{1}} \\ \hbar R_{\bar{1}}^{\bar{2}\bar{1}} & 1 + \hbar R_{\bar{2}}^{\bar{2}\bar{1}} & \hbar R_{\bar{2}}^{\bar{2}\bar{1}} \\ \hbar R_{\bar{1}}^{\bar{2}\bar{2}} & \hbar R_{\bar{2}}^{\bar{2}\bar{2}} & 2 + \hbar R_{\bar{2}}^{\bar{2}\bar{2}} \end{pmatrix}
\end{aligned}$$

then Proposition 4.2 is proved.

References

- [1] A. P. Balachandran, B. P. Dolan, J. -H. Lee, X. Martin and D. O'Connor, "Fuzzy complex projective spaces and their star products," *J. Geom. Phys.* **43**, 184 (2002) [hep-th/0107099].
- [2] Brian P. Dolan, Oliver Jahn "Fuzzy Complex Grassmannian Spaces and their Star Products," [hep-th/0111020].
- [3] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, "Deformation Theory And Quantization. 1. Deformations Of Symplectic Structures," *Annals Phys.* **111** (1978) 61.
F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, "Deformation Theory And Quantization. 2. Physical Applications," *Annals Phys.* **111** (1978) 111.
- [4] P. Bieliavsky, S. Detournay and P. Spindel, "The Deformation quantizations of the hyperbolic plane," *Commun. Math. Phys.* **289**, 529 (2009) [arXiv:0806.4741 [math-ph]].
- [5] M. Bordemann, M. Brischle, C. mmrich, S. Waldmann, "Phase Space Reduction for Star-Products: An Explicit Construction for $\mathbb{C}P^n$," *Lett. Math. Phys.* **36** (1996), 357.
- [6] M. Cahen, S. Gutt, J. Rawnsley, "Quantization of Kahler manifolds, II," *Am. Math. Soc. Transl.* **337**, 73 (1993).
- [7] M. Cahen, S. Gutt, J. Rawnsley, "Quantization of Kahler manifolds, IV," *Lett. Math. Phys.* **34**, 159 (1995).
- [8] M. De Wilde, P. B. A. Lecomte, "xistence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds," *Lett. Math. Phys.* **7**, 487 (1983).
- [9] B. Fedosov, "A simple geometrical construction of deformation quantization," *J. Differential Geom.* **40**, 213 (1994).
- [10] M. B. Halima, Tilmann Wurzbacher, "Fuzzy complex Grassmannians and quantization of line bundles" *Semin. Univ. Hambg.* (2010) 80: 59.
- [11] M. B. Halima, "Construction of certain fuzzy flag manifolds" *Rev. Math. Phys.* **22**, 533 (2010)
- [12] K. Hayasaka, R. Nakayama and Y. Takaya, "A New noncommutative product on the fuzzy two sphere corresponding to the unitary representation of $SU(2)$ and the Seiberg-Witten map," *Phys. Lett. B* **553**, 109 (2003) [hep-th/0209240].
- [13] D. Karabali, V. P. Nair and S. Randjbar-Daemi, "Fuzzy spaces, the M(atrrix) model and the quantum Hall effect," In *Shifman, M. (ed.) et al.: From fields to strings, vol. 1* 831-875 [hep-th/0407007].

- [14] A. V. Karabegov, “On deformation quantization, on a Kahler manifold, associated to Berezin’s quantization,” *Funct. Anal. Appl.* **30**, 142 (1996).
- [15] A. V. Karabegov, “Deformation quantizations with separation of variables on a Kahler manifold,” *Commun. Math. Phys.* **180**, 745 (1996) [arXiv:hep-th/9508013].
- [16] A. V. Karabegov, “An explicit formula for a star product with separation of variables,” [arXiv:1106.4112 [math.QA]].
- [17] A. V. Karabegov, “PseudoKahler quantization on flag manifolds,” dg-ga/9709015.
- [18] Y. Kitazawa, “Matrix models in homogeneous spaces,” *Nucl. Phys. B* **642**, 210 (2002) [hep-th/0207115].
- [19] S. Kobayashi and K. Nomizu, “Foundation of Differential Geometry, volume II,” John Wiley and Sons, Inc , 1969
- [20] M. Kontsevich, “Deformation quantization of Poisson manifolds, I,” *Lett. Math. Phys.* **66**, 157 (2003) [arXiv:q-alg/9709040].
- [21] Y. Maeda, A. Sako, T. Suzuki and H. Umetsu, “Deformation Quantization with Separation of Variables and Gauge Theories,” *Proceedings, 33th Workshop on Geometric Methods in Physics (XXXIII WGMP) : Bialowieza, Poland, June 29-July 5, 2014* ,p.135-144
- [22] C. Moreno, “*-products on some Kähler manifolds”, *Lett. Math. Phys.* **11**, 361 (1986).
- [23] C. Moreno, “Invariant star products and representations of compact semisimple Lie groups,” *Lett. Math. Phys.* **12**, 217 (1986).
- [24] S. Murray and C. Saemann, “Quantization of Flag Manifolds and their Supersymmetric Extensions,” *Adv. Theor. Math. Phys.* **12** (2008) no.3, 641 doi:10.4310/ATMP.2008.v12.n3.a5 [hep-th/0611328].
- [25] T. Ohsaku, “Algebra of noncommutative Riemann surfaces,” math-ph/0606057.
- [26] H. Omori, Y. Maeda, and A. Yoshioka, “Weyl manifolds and deformation quantization,” *Adv. in Math.* **85**, 224 (1991).
- [27] A. Sako, T. Suzuki and H. Umetsu, “Explicit Formulas for Noncommutative Deformations of CP^N and CH^N ,” *J. Math. Phys.* **53**, 073502 (2012) [arXiv:1204.4030 [math-ph]].
- [28] A. Sako, T. Suzuki and H. Umetsu, “Noncommutative CP^N and CH^N and their physics,” *J. Phys. Conf. Ser.* **442**, 012052 (2013).
- [29] A. Sako, T. Suzuki and H. Umetsu, “Gauge theories on noncommutative CP^N and Bogomolnyi-Prasad-Sommerfield-like equations,” *J. Math. Phys.* **56**, no. 11, 113506 (2015). [arXiv:1506.06957[hhep-th]]
- [30] A. Sako and H. Umetsu, “Twisted Fock Representations of Noncommutative Kähler Manifolds,” arXiv:1605.02600 [math-ph].

- [31] M. Schlichenmaier, “Berezin-Toeplitz quantization and star products for compact Kähler manifolds,” *Contemp. Math.* **583** (2012) 257
- [32] M. Schlichenmaier, “Some naturally defined star products for Kähler manifold,” *Trav. math.* **20** (2012) 187